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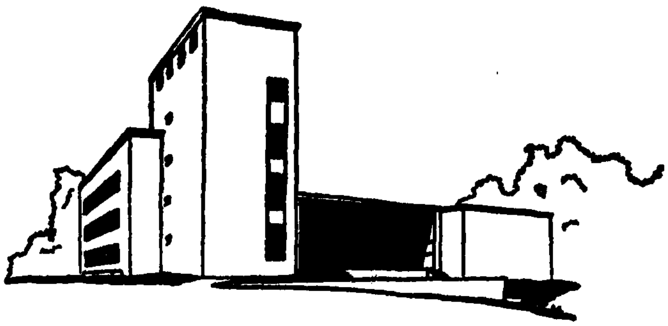

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by

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ON QUADRATIC PROGRAMS WITH A SINGLE EQUALITY CONSTRAINT

Jong-Shi Pang

Abstract. This paper shows that an alogrithm developed by the author in an earlier paper for solving singly constrained quadratic programs is polynomially bounded in the number of variables of the program if the objective function has non-positive mixed second derivatives.

Key Words. Quadratic program, polynomially bounded, algorithm, decomposition, parametric linear complementarity.

1. Introduction

Recently, there have been several papers dealing with convex quadratic programs having only upper and lower bounds on the variables and one single equality constraint [7, 9, 10, 11, 15]. Quadratic programs of this kind have applications in many different areas; see the cited papers for references. In these papers, a common and rather interesting approach was suggested for the solution of such quadratic programs. The approach can be briefly outlined as follows. In the Karush-Kuhn-Tucker optimality conditions of a given program, the Lagrange multiplier associated with the equality constraint is treated as a parameter and not as a variable of the problem. Then, ignoring the equality constraint, the remaining conditions constitute a parametric linear complementarity problem. From parametric linear complementarity theory, it is known that the solution to the resulting complementarity problem is a piecewise linear function of the parameter. The problem of solving the original program therefore reduces to finding a suitable value of the parameter for which the corresponding solution to the (parametric) linear complementarity problem also satisfies the outstanding equality constraint. Incidentally, this approach may be considered as a special application of the <u>decomposition principle</u> for general convex programs described in Rockafellar [18].

In [7, 9, 10, 11], by taking advantage of some special properties of the objective function (separability, e.g.), the resulting parametric linear complementarity problem (and therefore the original quadratic program) can be solved extremely easily. In fact, as pointed out in [11], the total amount of computational effort (i.e., additions, multiplications and comparisons) required in the separable case is bounded above by a low-degree polynomial in the size of the given program.

Based on the technique of parametric principal pivoting and the decomposition approach mentioned above, the author [15] has developed a general algorithm for solving the single constrained strictly convex quadratic program

$$\text{minimize} \quad q^T x + \tfrac{1}{2} x^T Q x \quad \text{subject to} \quad c^T x = d \quad \text{and} \quad 0 \leqq x \leqq a \qquad (1)$$

where the matrix Q is symmetric positive definite, the vectors a and c and the scalar d are all positive. Our purpose in this paper is to show that the cited algorithm is polynomially bounded in the order of the matrix Q if Q is a <u>Stieltjes matrix</u>, i.e., if Q in addition to being symmetric positive definite has all *off-diagonal* entries nonpositive. Note that a diagonal matrix Q which yields a separable objective function is obviously Stieltjes. Some other related references for the quadratic program (1) are [1, 4, 5, 12, 14, 16, 19].

2. The Main Result

We find it useful to briefly review the algorithm described in [15] for solving the quadratic program (1). We may write the Karush-Kuhn-Tucker optimality conditions for the program as

$$u = q + tc + Qx + y \geqq 0, \quad x \geqq 0, \quad u^T x = 0 \tag{2i}$$

$$v = a - x \geqq 0, \quad y \geqq 0, \quad v^T y = 0 \tag{2ii}$$

$$d = c^T x \tag{2iii}$$

where t is the Lagrange multiplier associated with the equality constraint (2iii). The conditions (2i) and (2ii) define a parametric linear complementarity problem (with t as the parameter) to which the parametric principal pivoting algorithm [3] is applicable. With this latter algorithm, a solution function $x^*(t)$ can be computed. (Several simplifications can be made in the application of this pivoting algorithm. For instance, it can be shown that the 2×2 block pivots will never take place [12, 15].) The search for a suitable t^* such that $x^*(t^*)$ satisfies (2iii) as well can be achieved by a simple interpolation scheme. If the quadratic program (1) is feasible, then the existence of t^* (and therefore the success of the above procedure) is ensured by the positive definiteness of Q.

The requirement that the vector c be stricly positive is useful in order to initiate the parametric principal pivoting algorithm. It can be replaced by the weaker assumption that c be merely nonnegative with the provision that there is a $\bar{t}$ such that $q + tc$ is nonnegative for all $t \geqq \bar{t}$.

It is well-known that the solution $x^*(t)$ is a piecewise linear function of t. Therefore, so is $f(t) = c^T x^*(t) - d$. Basically, the interpolation step is to find out which segments of linearity of $f(t)$ contains its zero. The search is carried out sequentially from one segment to the next, starting from the infinite interval $[t_1, \infty)$ where t_1 is the first <u>critical value</u> of t, i.e., the first breakpoint of $f(t)$ from the right. Obviously, given the value of $x^*(t)$ in a segment of linearity, the amount of computational effort required in such interpolation is linear.

The movement from one segment to another is accomplished by *principal pivoting*. In order to show that the overall procedure is polynomially bounded if Q is a Stieltjes matrix, it suffices to prove that the number of necessary pivots is polynomially bounded. As a matter of fact, we show that this number is at most $2n$ where n is the order of Q. The first step to establish this assertion is to note that each pivot changes the status of an x-variable in four possible ways: (i) from nonbasic at lower bound to basic, (ii) from basic to nonbasic at upper bound, (iii) from nonbasic at upper bound back to basic, and (iv) from basic back to nonbasic at lower bound. Note that it is not possible to change directly from nonbasic at either bound to nonbasic at the other bound. Such a change requires two pivots. The following theorem provides the key to establish to desired polynomial boundedness.

<u>Theorem</u>. Let Q be a Stieltjes matrix. Then in the application of the parametric principal pivoting algorithm to solve the parametric linear complementarity problem defined by (2i) and (2ii), each pivot must correspond to either the change of a nonbasic x-variable at lower bound becoming basic or that of a basic x-variable becoming nonbasic at upper bound.

Althernatively stated, the theorem says that if an x-variable has become basic, it can never become nonbasic at zero again, and if an x-variable has reached its upper bound, it will stay there through termination of the algorithm. If not for the degenerate pivots, the theorem can be proved easily by observing the fact that the solution $x^*(t)$ is a nondecreasing functon of t. This latter fact follows from a least-element characterization of $x^*(t)$ [14]. In what follows, we give a direct proof of the theorem.

<u>Proof of Theorem</u>. Let I_1 and I_2 be the index sets of the x-variables that are currently basic and nonbasic at upper bounds, respectively. Let J be the complement of I_1 union I_2. Consider the canonical system of the parametric linear complementarity problem (defined by (2i) and (2ii)) with respect to these index sets. In the system, the constant and parametric vectors are given by

Basic Variables		Constant Column	Parametric Column (t)	
x_{I_1}		$\bar{q}_{I_1}$	$\bar{c}_{I_1}$	
y_{I_2}		$\bar{q}_{I_2}$	$\bar{c}_{I_2}$	
u_J	=	$\bar{q}_J$	$\bar{c}_J$	Nonbasic Portion
v_{I_1}		$a_{I_1} - \bar{q}_{I_1}$	$-\bar{c}_{I_1}$	
v_J		a_J	$\emptyset$	
x_{I_2}		a_{I_2}	$\emptyset$	

where

$$(\bar{q}_{I_1}, \bar{c}_{I_1}) = -(Q_{I_1 I_1})^{-1}(q_{I_1} + Q_{I_1 I_2} a_{I_2}, c_{I_1})$$

$$(\bar{q}_{I_2}, \bar{c}_{I_2}) = -(q_{I_2} + Q_{I_2 I_2} a_{I_2}, c_{I_2}) - Q_{I_2 I_1}(\bar{q}_{I_1}, \bar{c}_{I_1})$$

$$(\bar{q}_J, \bar{c}_J) = (q_J + Q_{J I_2} a_{I_2}, c_J) + Q_{J I_1}(\bar{q}_{I_1}, \bar{c}_{I_1}).$$

To determine the next pivot, the following ratio test is performed,

$$\max\{\max\{-\bar{q}_i/\bar{c}_i : \bar{c}_i > \emptyset\}, \max\{(a_i - \bar{q}_i)/\bar{c}_i : i \text{ in } I_1, \bar{c}_i < \emptyset\}\}.$$

If k is a maximizing index, then depending on where k comes from, a simple principal pivot is performed.

Since $c_{I_1} \geqq \emptyset$ by assumption and since $Q_{I_1I_1}$ has a non-negative inverse [8], it follows that $\bar{c}_{I_1} = -(Q_{I_1I_1})^{-1}c_{I_1} \leqq \emptyset$. Therefore, we have

$$\bar{c}_{I_2} = -c_{I_2} - Q_{I_2I_1}\bar{c}_{I_1} \leqq \emptyset$$

because $Q_{I_2I_1} \leqq \emptyset$. Consequently, the maximizing index k belongs to either J or I_1 and the next pivot can occur only at either a u_J-row or a v_{I_1}-row. If k is in J, then the corresponding x_k-variable is becoming basic; whereas if k is in I_1, then x_k is reaching its upper bound. Since a pivot will never occur at a x_{I_1} or a y_{I_2}-row, we have established the theorem.

If we let I_1 and I_2 be the index sets as defined in the above proof, then after each pivot, either an index k is transferred from I_1 to I_2, in which case the cardinality of I_1 decreases by 1 and that of I_1 union I_2 remains unchanged; or else an index k not in I_1 and I_2 is added to I_1, in which case the cardinality of I_1 (and I_1 union I_2) increases by 1. Since n is the number of x-variables, it follows that after at most 2n pivots, the algorithm must terminate. This completes the proof of our claim that the solution procedure described above for solving the quadratic program (1) is polynomially bounded if Q is a Stieltjes matrix. Finally, since we base our argument on a monotonicity property of index sets, the proof is valid under absolutely no nondegeneracy assumption.

3. Some Concluding Remarks

There have been several recent papers (see [2] and references therein) demonstrating how Khachian's ellipsoidal algorithm for linear programming [13] can be extended to solve general convex quadratic programs. Although such ellipsoidal algorithms are polynomially bounded, computational experience [6] have shown clearly that they are at their present stage, far from being competitive with some pivoting methods for solving practical problems of considerable size.

The algorithm discussed in the last section is of an entirely different category. On the one hand, computational experience [17] has shown that the algorithm performs fairly well on large problems. On the other hand, by some simple operation count, one can show easily that the total computational effort required is at most of the order n^4. This is significantly less than that required by the ellipsoidal algorithms.

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